and therefore

$$\frac{e}{\hbar c}\int B_{[\mu\nu]} \quad dS^{\mu\nu} = 2\pi \sum n_s e_s \, .$$

Again the integral represents magnetic charge, which is now the matrix

$$Q_m = (\hbar c/e) 2\pi \sum n_s e_s \, .$$

The eigenvalues of Q_m are

$$Q'_m = (\hbar c/e) 2\pi \sum n_s e'_s \qquad (13.12)$$

where $e'_s = 1,0$. Equation (13.12) is the generalization of the Dirac condition to the case of an arbitrary gauge group.[8]

14. MAGNETIC POLES

We have seen that the Yang–Mills field whose neutral component is the electromagnetic field does not admit magnetic poles. However there is also a theory in which the neutral component is dual to the usual electromagnetic field. In the one case the charged bosons carry electric charge, magnetic moment, and no magnetic charge. In the dual theory they carry magnetic charge, electric moment, and no electric charge. In the latter case the fermion sources are also magnetic monopoles, and the magnetic charge is always given by (13.12). If both classes of particles exist the observed electric (magnetic) fields are due to electric (magnetic) monopoles at rest and magnetic (electric) monopoles in motion.

[8] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 1 (1962).

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The Vacuum Trajectory in Conventional Field Theory*

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1. INTRODUCTION

It has recently been shown that the conventional field theory of vector bosons (γ) interacting through a conserved current with spin one-half fermions (N) (i.e. quantum electrodynamics with massive photons) possesses several remarkable and hitherto unexpected properties. First, it appears to have finite nonperturbative solutions, as shown by Johnson, Baker, and Willey.[1] Second, as shown by Marx, Zachariasen, and the present authors,[2] the spin $\frac{1}{2}$ particle, which in second-order perturbation theory appears as a fixed singularity in the angular momentum, is, as a result of radiative corrections, found to lie on a Regge trajectory

$$l = J - \tfrac{1}{2} = \alpha(W) \, ,$$

where W is the total energy and $\alpha(m) = 0$, with m the fermion mass. The function $\alpha(W)$ has a power series expansion in the coupling constant, called γ, such that $\alpha(W) \sim \gamma^2$ as $\gamma \to 0$.

In this paper we investigate the generation of a Pomeranchuk-like (or P) trajectory in the same theory. This differs from the previously discussed fermion trajectory problem in several ways. The P trajectory in no way corresponds to an elementary particle of field theory but is more analogous to the well understood trajectories of potential theory (or ladder approximations in field theory) with the difference that it approaches $J = 1$ as $\gamma^2 \to 0$ rather than $J = -1$ as in potential theory. This is due to the spin of the particles involved. Another difference from the fermion trajectory is, as we shall see, that $J - 1 \equiv \Delta(W^2) \sim \gamma^4$ as $\gamma \to 0$.

The processes whose asymptotic behavior at large z is determined by the P trajectory are $\gamma + \gamma \to \gamma + \gamma$, $N + \overline{N} \to N + \overline{N}$ and $\gamma + \gamma \to N + \overline{N}$. The latter amplitude can be calculated in order γ^2, the two former ones in γ^4. The generation of a Regge

* Supported in part by the U. S. Atomic Energy Commission and the Air Force Office of Scientific Research.
[1] K. Johnson, M. Baker, and R. Willey, Phys. Rev. Letters **11**, 518 (1963).
[2] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen Phys. Rev. **133**, 145 (1964) (hereinafter referred to as III).

trajectory in perturbation theory inevitably requires first the existence of a "nonsense" channel at the relevant value of J, which in this case is the two boson ${}^5(J-2)_J$ state at $J=1$. The parity of this state is $(-1)^J$ where even values of J are physical. The second requirement, as was shown in Ref. 2, is the factorization of the Born approximation which holds for the above reactions, at least for the "nonsense" channel. This peculiar qualification results from the fact that an investigation of the "sense" boson states requires a higher order calculation ($\sim\gamma^6$) than we have performed for the reaction $\gamma+\gamma\to N+\overline{N}$. To order γ^2 it turns out that the "sense" amplitudes go like

$$P_J(z)+P_J(-z)\sim z-z=0\,,$$

$$P_J'(z)-P_J'(-z)\backsim 1-1=0\,,$$

for large z near $J=1$; this has the consequence that the coefficients of z and $-z$ cannot be separately and unambiguously calculated. The nonsense amplitude on the other hand goes like

$$P_J''(z)+P_J''(-z)\backsim 1/z-1/z$$

where the $1/z$ term comes from $1/m^2-t$ and the $-1/z$ term from $1/m^2-u$; these terms can be separately identified and their coefficients calculated.

We have used the same technique as in III to show that the existence of a perturbation theory vacuum trajectory is consistent with unitarity and analyticity. We have throughout considered the theory of spin $\frac{1}{2}$ fermions, which as stated above shows the factoring property which is necessary for the success of the Reggeization procedure. However, for the problem of the vacuum trajectory as opposed to that of the elementary nucleon discussed in III, the scalar nucleon theory appears to Reggeize as easily as the spin $\frac{1}{2}$ theory, since in the former theory the factorization requirement is empty (there being only one $N\overline{N}$ helicity state for spin zero nucleons).

Our final formula for the trajectory is

$$J=1+\Delta(W^2)\,,$$

where

$$\Delta(W^2)=Y\sum_i X_{ii}/\sum(\xi_\rho\xi_i)^2\,.$$

Here $\xi_\rho\xi_i$ is calculated from the second order $\gamma+\gamma\to N+\overline{N}$ amplitude, X_{ij} from the 4th order $N+\overline{N}\to N+\overline{N}$ amplitudes, and Y from the 4th order $\gamma+\gamma\to\gamma+\gamma$ amplitude, all at high z and all using only the right-hand (positive) z cut (the t cut, in other words). These quantities will be precisely defined later. The reason why one must discuss separately the positive and negative z cuts is given later.

2. DESCRIPTION OF THE STATES

The description of the relevant states of the $\gamma\gamma$ and $N\overline{N}$ systems is sufficiently involved that we shall give it in some detail. In order to keep track of all factors of two associated with particle identity one must be exceedingly careful.

We start by enumerating in Table I the states of

TABLE I. States of the $\gamma\gamma$ system.

State	Parity (P)	Value of J permitted by statistics
${}^5(J+2)_J$	$(-1)^J$	even
${}^5(J+1)_J$	$-(-1)^J$	odd
${}^5(J)_J$	$(-1)^J$	even
${}^5(J-1)_J$	$-(-1)^J$	odd
${}^5(J-2)_J$	$(-1)^J$	even
${}^3(J+1)_J$	$-(-1)^J$	even
${}^3(J)_J$	$(-1)^J$	odd
${}^3(J-1)_J$	$-(-1)^J$	even
${}^1(J)_J$	$(-1)^J$	even

the $\gamma\gamma$ system, together with their parity, and values of (physical) J allowed by statistics, using conventional L–S coupling notation. Note that the quintet spin states are symmetric under interchange of the γ's whereas the triplet states are odd (being essentially the components of the cross product of polarization vectors). These facts must be borne in mind when verifying Table I.

In Table II we give a similar enumeration of the $N\overline{N}$ system and limit our consideration to states with charge conjugation quantum number (C) equal to $+1$, because this is what we are ultimately interested in. Note that for the $N\overline{N}$ system the role played by the statistics for the $\gamma\gamma$ system in limiting physical J values is taken over by charge conjugation. In constructing Table II, one must remember that the intrinsic parity of particle and antiparticle is opposite and that the operation of charge conjugation carries with it an unusual minus sign associated with the anti-commutation of fermion field operators. Thus if $a_i^\dagger$ and $b_j^\dagger$ are particle and antiparticle creation operators the state $a_i^\dagger b_j^\dagger|0\rangle$ under charge conjugation goes into $b_i^\dagger a_j^\dagger|0\rangle=-a_j^\dagger b_i^\dagger|0\rangle$.

The content of Tables I and II may be conveniently summarized as is shown in Tables III and IV.

The number of transition amplitudes of definite J and P for the process $\gamma+\gamma\to N+\overline{N}$ is thus:

$$N=4\times 2(\text{even }J,\text{even }P)+2\times 1(\text{even }J,\text{odd }P)+2\times 1(\text{odd }J,\text{even }P)\,. \tag{2.1}$$

The total number of transition amplitudes with even J is therefore ten, and with odd J, two. This may be contrasted with the $\gamma + N \to \gamma + N$ case considered in III, where there are twelve amplitudes for J either even or odd. The reduction in number by a factor of two is a consequence of the identity and charge conjugation properties of the γ's.

Table II. States of the $N\bar{N}$ system.

State	Parity (P)	Charge Conjugation (C)	Value of J permitted by $C = +1$
$^3(J+1)_J$	$(-1)^J$	$(-1)^J$	even
$^3(J)_J$	$-(-1)^J$	$-(-1)^J$	odd
$^3(J-1)_J$	$(-1)^J$	$(-1)^J$	even
$^1(J)_J$	$-(-1)^J$	$(-1)^J$	even

Table III. $\gamma\gamma$-system: Number of states with given J and P.

	Even P	Odd P
Even J	4	2
Odd J	2	1

Table IV. $N\bar{N}$-system: Number of states with given J and P.

	Even P	Odd P
Even J	2	1
Odd J	1	0

It is useful for our purposes to express the enumeration of states in terms of the parity conserving helicity amplitudes used in III. The identity of particles causes certain complications which we must deal with, so we approach the problem gradually. We recall from Jacob and Wick[3] that the exchange of particles for pure helicity states is given by

$$P_{12}|JM;\lambda_1\lambda_2\rangle = (-1)^{J-s_1-s_2}|JM;\lambda_2\lambda_1\rangle\,, \quad (2.2)$$

and that the parity operator is

$$P|JM;\lambda_1\lambda_2\rangle = (-1)^{J-s_1-s_2}\eta_1\eta_2|JM;-\lambda_1\,-\lambda_2\rangle\,, \quad (2.3)$$

where s represents the particle spin and η its intrinsic parity. In our problem the γ's have $s = 1$, $\eta = -1$ and the nucleons have $s = \frac{1}{2}$, but for the $N\bar{N}$ system $\eta_1\eta_2 = -1$. Thus in either case $\eta_1\eta_2(-1)^{-s_1-s_2} = +1$. Furthermore the charge conjugation operator on a $2-\gamma$ state is $+1$ (hence our obsession with $C = +1$) whereas applied to a $N\bar{N}$ state, C is equivalent to $-P_{12}$, the minus sign being the same one discussed in connection with Table II.

[3] M. Jacob and G. C. Wick, Ann. Phys. **7**, 404 (1959).

In Tables V and VI we show the pure helicity states of the $\gamma\gamma$ and $N\bar{N}$ system. (We have not yet introduced the parity conserving helicity states.) In constructing Table V we must remember that we, of course, limit attention to states of the appropriate symmetry, namely (aside fro mnormalization), $|\lambda_1\lambda_2\rangle + P_{12}|\lambda_1\lambda_2\rangle$. If $\lambda_1 = \lambda_2$, we must have even J as follows instantly from Eq. (2.2). If $\lambda_1 \neq \lambda_2$, there is no restriction on J and one simply uses the symmetrized states $|\lambda_1\lambda_2\rangle + (-1)^J|\lambda_2\lambda_1\rangle$. Now consider the behavior of these under the parity operation. We have

Table V. $\gamma\gamma$-system: Independent helicity states.

Helicity State	Allowed J values	Parity reflection
$\vert 1\ \ 1\rangle$	even	$+\vert -1\ -1\rangle$
$\vert 1\ \ 0\rangle$	$\binom{\text{even}}{\text{odd}}$	$\pm\vert -1\ \ 0\rangle$
$\vert 1\ -1\rangle$	$\binom{\text{even}}{\text{odd}}$	$+\vert\ 1\ -1\rangle$
$\vert 0\ \ 0\rangle$	even	$+\vert\ 0\ \ 0\rangle$
$\vert -1\ \ 0\rangle$	$\binom{\text{even}}{\text{odd}}$	$\pm\vert -1\ \ 0\rangle$
$\vert -1\ -1\rangle$	even	$+\vert\ 1\ \ 1\rangle$

Table VI. $N\bar{N}$-system: Independent helicity states.

$N\bar{N}$	C-reflection	P-reflection
$\vert \frac{1}{2}\ \ \frac{1}{2}\rangle$	$(-1)^J\vert\ \frac{1}{2}\ \ \frac{1}{2}\rangle$	$(-1)^J\vert -\frac{1}{2}\ -\frac{1}{2}\rangle$
$\vert \frac{1}{2}\ -\frac{1}{2}\rangle$	$(-1)^J\vert -\frac{1}{2}\ \ \frac{1}{2}\rangle$	$(-1)^J\vert -\frac{1}{2}\ \ \frac{1}{2}\rangle$
$\vert -\frac{1}{2}\ \ \frac{1}{2}\rangle$	$(-1)^J\vert\ \frac{1}{2}\ -\frac{1}{2}\rangle$	$(-1)^J\vert\ \frac{1}{2}\ -\frac{1}{2}\rangle$
$\vert -\frac{1}{2}\ -\frac{1}{2}\rangle$	$(-1)^J\vert -\frac{1}{2}\ -\frac{1}{2}$	$(-1)^J\vert\ \frac{1}{2}\ \ \frac{1}{2}\rangle$

$$P\{|\lambda_1\lambda_2\rangle + (-1)^J|\lambda_2\lambda_1\rangle\} = (-1)^J|-\lambda_1\,-\lambda_2\rangle + |-\lambda_2\,-\lambda_1\rangle\,.$$

In the interesting cases we have either $\lambda_1 = \pm 1$, $\lambda_2 = 0$ or $\lambda_1 = 1$, $\lambda_2 = -1$. In the former case we find

$$P\{|\pm 1\ 0\rangle + (-1)^J|0\pm 1\rangle\} = (-1)^J\{|\mp 1\ 0\rangle + (-1)^J|0\mp 1\rangle\}\,,$$

hence the one state goes into the other with a plus or minus sign depending on J being even or odd. For the case $\lambda_1 = 1$, $\lambda = -1$, however,

$$P\{|1-1\rangle + (-1)^J|-1\ 1\rangle\} = +\{|1-1\rangle + (-1)^J|-1\ 1\rangle\}\,,$$

for both even *and* odd J. The verification of Table VI is simpler; one need only remember that $s_1 = s_2 = \frac{1}{2}$, $\eta_1\eta_2(-1)^{s_1 s_2} = +1$ and $C = -P_{12}$.

The eigenfunctions of $C = 1$ and P for the $\gamma\gamma$ and $N\bar{N}$ systems are shown in Tables VII and VIII. The asterisk in Table VII labels the one state of the $\gamma\gamma$ system which does not combine with the $C = 1N\bar{N}$ system. We show in both cases the parity conserving

TABLE VII. $\gamma\gamma$-system: Parity conserving helicity states, $C = +1$.

State	J	P
$\vert 1\ \ 1\rangle_+$	even	$+$
$\vert 1\ \ 1\rangle_-$	even	$-$
$\vert 1\ \ 0\rangle_+$	$\binom{\text{even}}{\text{odd*}}$	$\binom{+}{-}$
$\vert 1\ \ 0\rangle_-$	$\binom{\text{even}}{\text{odd}}$	$\binom{-}{+}$
$\vert 0\ -1\rangle_+$	even	$+$
$\vert 1\ -1\rangle_+$	even	$+$
$\vert 1\ -1\rangle_-$	odd	$+$

TABLE VIII. $N\bar{N}$-system: Parity conserving helicity states, $C = +1$.

State	J	P
$\vert \frac{1}{2}\ \ \frac{1}{2}\rangle_+$	even	$+$
$\vert \frac{1}{2}\ \ \frac{1}{2}\rangle_-$	even	$-$
$\vert \frac{1}{2}\ -\frac{1}{2}\rangle_+$	even	$+$
$\vert \frac{1}{2}\ -\frac{1}{2}\rangle_-$	odd	$+$

helicity states which are defined in III, Eq. (2.3) which for the present discussion reduces to

$$|JM;\lambda_1\lambda_2\rangle_\pm = \tfrac{1}{2}\{|JM;\lambda_1\lambda_2\rangle \pm |JM;-\lambda_1\ -\lambda_2\rangle\}$$
$$= \tfrac{1}{2}\{|JM;\lambda_1\lambda_2\rangle \pm (-1)^J P|JM;\lambda_1\lambda_2\rangle\}\ ,$$

since we are concerned here with only integral J's. It can now be seen that the counting of states with definite J and P in Tables VII and VIII is identical to that shown in Tables III and IV. There are again in the parity conserving helicity representation, in the reaction $\gamma + \gamma \rightarrow N + \bar{N}$ eight amplitudes (J even, $P+$), two amplitudes (J even, $P-$), and two amplitudes (J odd, $P+$), just as in Eq. (2.1).

As we have remarked, we deal with symmetrized two γ states and we must now see how this symmetrization affects our formulas for the full scattering amplitude.[4] From Ref. 3, Eq. (45) we construct the appropriate plane wave state (for our problem where $s_1 = s_2 = 1$)

$$|\mathbf{p};\lambda_1\lambda_2\rangle = \frac{1}{\sqrt{2}}\{|\mathbf{p};\lambda_1\lambda_2\rangle + (-1)^{\lambda_2-\lambda_1}e^{i\pi J_y}|\mathbf{p};\lambda_2\lambda_1\rangle\}\ . \tag{2.8}$$

[4] This part of the paper is a must for readers who delight in finding errors in factors of two. It can be read rather briefly by others.

For ease of writing, we consider the reaction $N\bar{N}\gamma + \gamma$ and find for the symmetrized scattering amplitude

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \frac{1}{\sqrt{2}}\{f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\theta) + (-1)^{-\lambda_c+\lambda_d}f_{\lambda_d\lambda_c;\lambda_a\lambda_b}(\theta-\pi)\} \tag{2.9}$$

To simplify this expression we express the characteristic functions $d^J_{\lambda\mu}$ (see Ref. 3) by the simpler functions $e^J_{\lambda\mu}$ defined in Ref. 2:

$$d^J_{\lambda\mu}(\theta) = (\sqrt{2}\sin\tfrac{1}{2}\theta)^{|\lambda-\mu|}(\sqrt{2}\cos\tfrac{1}{2}\theta)^{|\lambda+\mu|}e^J_{\lambda\mu}(z) \tag{2.10}$$

where $z = \cos\theta$. We write $\lambda = \lambda_a - \lambda_b$, $\mu = \lambda_c - \lambda_d$, and define

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) = (\sqrt{2}\sin\tfrac{1}{2}\theta)^{-|\lambda-\mu|} \times (\sqrt{2}\cos\tfrac{1}{2}\theta)^{-|\lambda+\mu|}f_{\lambda_c\lambda_d;\lambda_a\lambda_b}\ . \tag{2.11}$$

Then

$$f^S_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) = \frac{1}{\sqrt{2}}\{f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) + (-1)^{-\mu+|\lambda+\mu|}f_{\lambda_d\lambda_c;\lambda_a\lambda_b}(-z)\}\ . \tag{2.12}$$

Now the partial wave expansions of the f's are

$$f_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) = (k/p)^{\frac{1}{2}}\sum_J(2J+1)F^J_{\lambda_c\lambda_d;\lambda_a\lambda_b}e^J_{\lambda\mu}(z)\ ,$$
$$f_{\lambda_d\lambda_c;\lambda_a\lambda_b}(-z) = (k/p)^{\frac{1}{2}}\sum_J(2J+1)F^J_{\lambda_d\lambda_c;\lambda_a\lambda_b}e^J_{\lambda-\mu}(-z)\ , \tag{2.13}$$

where $k(p)$ is the center of mass momentum of the $\gamma\gamma(N\bar{N})$ system, and the F^J are the correctly normalized partial wave amplitudes defined in Ref. 2. Defining

$$(-1)^{-\mu+|\lambda+\mu|} = (-1)^{\mu-|\lambda+\mu|} \equiv (-1)^\sigma \tag{2.14}$$

(since λ,μ are integral in this case), and writing m for the ordered pair of helicities $\lambda_c\lambda_d$, $\tilde{m}$ for $\lambda_d\lambda_c$ and l for $\lambda_a\lambda_b$ we find

$$f^S_{m;l}(z) = \frac{1}{\sqrt{2}}(k/p)^{\frac{1}{2}}\sum_J(2J+1) \times\left\{\frac{e^J_{\lambda\mu}(z)+(-1)^\sigma e^J_{\lambda-\mu}(-z)}{\sqrt{2}}\,\frac{F^J_{m;l}+F^J_{\tilde{m};l}}{\sqrt{2}} + \frac{e^J_{\lambda\mu}(z)-(-1)^\sigma e^J_{\lambda-\mu}(-z)}{\sqrt{2}}\,\frac{F^J_{m;l}-F^J_{\tilde{m};l}}{\sqrt{2}}\right\}. \tag{2.15}$$

Now since the symmetrized two γ state is

$$|\lambda_1\lambda_2\rangle = \frac{1}{\sqrt{2}}\{|\lambda_1\,\lambda_2 + (-1)^J|\lambda_2\lambda_1\rangle\}\ , \tag{2.16}$$

we see that the symmetrized F-matrix element is

$$F^{JS}_{m;l} = \frac{1}{\sqrt{2}}[F^J_{m;l} + (-1)^J F^J_{\tilde{m};l}\ , \tag{2.17}$$

so that the physical, symmetrized matrix elements are

$$F^{JS}_{m;l} = \tfrac{1}{\sqrt{2}}\,[F^{J}_{m;l} + F^{J}_{\tilde{m};l}]\,, \quad J \text{ even}\,,$$
$$= \tfrac{1}{\sqrt{2}}\,[F^{J}_{m;l} - F^{J}_{\tilde{m};l}]\,, \quad J \text{ odd}\,. \qquad (2.18)$$

Using the results given in Appendix A of Ref. 2, we find

$$e^{J}_{\lambda-\mu}(-z) = (-1)^{J+\lambda} e^{J}_{\lambda\mu}(z)\,. \qquad (2.19)$$

We note that

$$\begin{aligned} e^{J}_{\lambda\mu}(z) \pm (-1)^{\sigma} e^{J}_{\lambda-\mu}(-z) &= e^{J}_{\lambda\mu}(z)[1 \pm (-1)^{J+\lambda+\mu-|\lambda+\mu|}] \\ &= e^{J}_{\lambda\mu}(z)[1 \pm (-1)^{J}] \end{aligned} \qquad (2.20)$$

[since $(-1)^{n-|n|} = +1$ for n a positive or negative integer]. It is gratifying, but hardly surprising, that our formalism has led us to the conclusion that only the physical, symmetrized partial wave amplitudes enter the expression for f:

$$f^{S}_{m;l}(z) = (k/p)^{\frac{1}{2}} \sum_{J} (2J+1) F^{JS}_{m;l} e^{J}_{\lambda\mu}(z)\,, \qquad (2.21)$$

where $F^{JS}_{m;l}$ was defined earlier, Eq. (2.17).

The final step in the procedure is the construction of the parity conserving scattering amplitudes according to Eq. (2.7) of Ref. 2 which may be written in the present notation as

$$f^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) = f^{S}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) \pm (-1)^{\lambda+\lambda_m} f^{S}_{-\lambda_c-\lambda_d;\lambda_a\lambda_b}(z)\,, \qquad (2.22)$$

where $\lambda_m = \max(|\lambda|,|\mu|)$. There comes now another nightmarish encounter with factors of $\sqrt{2}$ in order to insure that the normalization of the $F^{J\pm}$ constructed from our rule

$$F^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} = F^{JS}_{\lambda_c\lambda_d;\lambda_a\lambda_b} \pm F^{JS}_{-\lambda_c-\lambda_d;\lambda_a\lambda_b} \qquad (2.23)$$

has been correctly considered. Taking into account all peculiarities of the formalism occasioned by the symmetry of the γ's such as with $\lambda_c = \lambda_d$, $F^{JS}_{m;l} = 0$ for odd J and is $\sqrt{2}\,F^{J}_{m;l}$ for even J, etc. we conclude finally

$$f^{\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) = \sqrt{2}(k/p)^{\frac{1}{2}} \sum_{J} (2J+1) e^{J+}_{\lambda\mu}(z) F^{J\pm}_{\lambda_c\lambda_d;\lambda_a\lambda_b} + e^{J-}_{\lambda\mu}(z) F^{J\mp}_{\lambda_c\lambda_d;\lambda_a\lambda_b}\} \qquad (2.24)$$

and the sum extends over even or odd integers; precisely which combinations survive can be deduced by reference to Tables VII and VIII. With the explicitly written $\sqrt{2}$ all the $F^{\pm}$ may be taken to have the standard normalization adopted in Ref. 2.

We shall not give the correspondingly ghastly discussion of the process $\gamma + \gamma \to \gamma + \gamma$ or the quite trivial case of $N\overline{N} \to N\overline{N}$. In the next section we simply state the results for the cases of interest here.

3. THE RELEVANT PARTIAL WAVE EXPANSIONS

We are concerned with the trajectory with quantum numbers $C = +1$, $P = +1$, which is physical for even J. It is therefore (c.f. Tables VII and VIII) associated with the $|\frac{1}{2}\,\frac{1}{2}\rangle_+$ and $|\frac{1}{2}\,-\frac{1}{2}\rangle_+$ $N\overline{N}$ states and with the $|1\,1\rangle_+$, $|1\,0\rangle_+$, $|0\,0\rangle_+$ and $|1\,-1\rangle_+$ $\gamma\gamma$ states. As in III, our method of calculation will consist essentially of evaluating the appropriate f amplitudes at large z and extracting from them, by formal projection or informal inspection (in practice) of the behavior of the F^{J+} amplitudes in the neighborhood of the conjectured singularity in the neighborhood of $J = 1$. Since $e^{J+}_{\lambda\mu}$ dominates $e^{J-}_{\lambda\mu}$ at large z it is sufficient to calculate the scattering amplitude $f^{+}_{\lambda_c\lambda_d;\lambda_a\lambda_b}$ as $z \to \infty$ and deduce the values of the F^{J+} therefrom.

For the reaction $N + \overline{N} \to \gamma + \gamma$ the required relations are (always at large z)

$$\begin{aligned}
f^{+}_{0\,0;\frac{1}{2}\,\frac{1}{2}} &\cong \sqrt{2}(k/p)^{\frac{1}{2}} \sum_{J \text{ even}} (2J+1) F^{J+}_{0\,0;\frac{1}{2}\,\frac{1}{2}} e^{J+}_{0\,0}\,, \\
f^{+}_{1\,0;\frac{1}{2}\,\frac{1}{2}} &\cong \sqrt{2}(k/p)^{\frac{1}{2}} \sum_{J \text{ even}} (2J+1) F^{J+}_{1\,0;\frac{1}{2}\,\frac{1}{2}} e^{J+}_{0\,1}\,, \\
f^{+}_{1\,1;\frac{1}{2}\,\frac{1}{2}} &\cong \sqrt{2}(k/p)^{\frac{1}{2}} \sum_{J \text{ even}} (2J+1) F^{J+}_{1\,1;\frac{1}{2}\,\frac{1}{2}} e^{J+}_{0\,0}\,, \\
f_{1\,-1;\frac{1}{2}\,\frac{1}{2}} &\cong \sqrt{2}(k/p)^{\frac{1}{2}} \sum_{J \text{ even}} (2J+1) F^{J+}_{1\,-1;\frac{1}{2}\,\frac{1}{2}} e^{J+}_{0\,2}\,, \\
f^{+}_{0\,0;\frac{1}{2}\,-\frac{1}{2}} &\cong \sqrt{2}(k/p)^{\frac{1}{2}} \sum_{J \text{ even}} (2J+1) F^{J+}_{0\,0;\frac{1}{2}\,-\frac{1}{2}} e^{J+}_{1\,0}\,, \\
f^{+}_{1\,0;\frac{1}{2}\,-\frac{1}{2}} &\cong \sqrt{2}(k/p)^{\frac{1}{2}} \sum_{J \text{ even}} (2J+1) F^{J+}_{1\,0;\frac{1}{2}\,-\frac{1}{2}} e^{J+}_{1\,1}\,, \\
f_{1\,1;\frac{1}{2}\,-\frac{1}{2}} &\cong \sqrt{2}(k/p)^{\frac{1}{2}} \sum_{J \text{ even}} (2J+1) F^{J+}_{1\,1;\frac{1}{2}\,-\frac{1}{2}} e^{J+}_{1\,0}\,, \\
f_{1\,-1;\frac{1}{2}\,-\frac{1}{2}} &\cong \sqrt{2}(k/p)^{\frac{1}{2}} \sum_{J \text{ even}} (2J+1) F^{J+}_{1\,-1;\frac{1}{2}\,-\frac{1}{2}} e^{J+}_{1\,2}\,.
\end{aligned} \qquad (3.1)$$

The sign $\cong$ signifies equality after the large z limit.

We assume, as usual, that the functions F^{J+} can be continued to odd values of J. We may therefore replace

$$f^{+} \cong \sqrt{2}(k/p)^{\frac{1}{2}} \sum_{J \text{ even}} F^{J+} e^{J+} \qquad (3.2a)$$

by

$$f^{+} \cong \tfrac{1}{\sqrt{2}}\,(k/p)^{\frac{1}{2}} [\sum_{J} F^{J+} e^{J+}(z) \pm \sum_{J} F^{J+} e^{J+}(-z)]\,, \qquad (3.2b)$$

the $\pm$ sign depending on the reflection property of the e^{J+} involved. [We recall from III that $e^{J+}_{\lambda}(-z) = (-1)^{\lambda+\lambda_m} e^{J}_{\lambda}(z)$ where λ_m is the max $(|\mu|,|\lambda|)$]. The two terms evidently correspond to a decomposition

$$f^{+}(z) = f^{+}_{R}(z) + f^{+}_{L}(z)\,, \qquad (3.3)$$

where $f_R^{\pm}$ has (when the sum over J is replaced by a Watson–Sommerfeld integral) only a right-hand cut in z and $f_L^{\pm}$ only a left-hand cut, and $f_R^{\pm}(-z) = \pm f_L^{\pm}(z)$.

This is most easily seen by noting that, aside from subtractions,

$$f^{+}(z) = \int \frac{dz'\,\rho(z')}{z' - z} \pm \int \frac{dz'\,\rho(z')}{z' + z}. \qquad (3.4)$$

Because $f^{+}(z)$ is either an even or an odd function of z, the first term *or* the second term gives correctly all the F^{J+}'s with even J according to the relation

$$F^{J+} = \int dz'\, Q_J(z')\rho(z')$$

with suitably generalized Q_J (see Ref. 2, Appendix). This separation of the cuts is crucial for our perturbation theoretic investigation of the singularity in the J-plane near $J = 1$, since for in any finite order of perturbation theory the contribution to f^{+} from $J = 1$ precisely cancels, that value of J not being physical. We must obviously get started somehow and this is done by seperating the cuts.

We consider next $N\bar{N}$ scattering for which the transitions between the states of interest are described by

$$\begin{aligned}
f^{+}_{\frac12\,\frac12;\frac12\,\frac12} &\cong \sum_{J\text{ even}} (2J+1)F^{J+}_{\frac12\,\frac12;\frac12\,\frac12}\, e^{J+}_{0\,0}\,,\\
f^{+}_{\frac12\,-\frac12;\frac12\,\frac12} &\cong \sum_{J\text{ even}} (2J+1)F^{J+}_{\frac12\,-\frac12;\frac12\,\frac12}\, e^{J+}_{0\,1}\,,\\
f^{+}_{\frac12\,-\frac12;\frac12\,-\frac12} &\cong \sum_{J\text{ even}} (2J+1)F^{J+}_{\frac12\,-\frac12;\frac12\,-\frac12}\, e^{J+}_{1\,1}\,. \qquad (3.5)
\end{aligned}$$

These, of course, must also be split into right- and left-hand cut contributions.

Finally, for reasons to be described later, in this paper we are interested in $\gamma\gamma$ scattering only in the state which is nonsensical at $J = 1$, that is $|1\ -1\rangle_{+}$, and the corresponding amplitude is

$$f^{+}_{1\,-1;1\,-1} \cong 2 \sum_{J\text{ even}} (2J+1)F^{J+}_{1\,-1;1\,-1}\, e^{J+}_{2\,2}\,. \qquad (3.6)$$

4. REGGEIZATION OF THE $N + \bar{N} \to \gamma + \gamma$ AMPLITUDES

We now make the assumption that there exists a Regge pole with a trajectory given by $\alpha(W^2) = 1 + \Delta(W^2)$ where $\Delta(W^2)$ vanishes as the coupling constant $\gamma \to 0$. For such a pole in the neighborhood of $J = 1$ we write the amplitudes F^{J+} in the form

$$\begin{aligned}
&F^{J}_{0\,0;\frac12\,\pm\frac12} \cong \frac{\xi_{0\,0}\xi_{\frac12\,\pm\frac12}}{J-\alpha}\,, \quad F^{J}_{1\,0;\frac12\,\pm\frac12} \cong \frac{\xi_{1\,0}\xi_{\frac12\,\pm\frac12}}{J-\alpha}\,,\\
&F^{J}_{1\,1;\frac12\,\pm\frac12} \cong \frac{\xi_{1\,1}\xi_{\frac12\,\pm\frac12}}{J-\alpha}\,, \quad \frac{F^{J}_{1\,-1;\frac12\,\pm\frac12}}{(J-1)^{\frac12}} \cong \frac{\xi_{1\,-1}\xi_{\frac12\,\pm\frac12}}{J-\alpha}\,. \qquad (4.1)
\end{aligned}$$

With these conjectured forms for the various F^{J+} we may compute the asymptotic limit of the f amplitudes for the process in the usual Regge–Watson–Sommerfeld way, in which

$$\begin{aligned}
\frac{d^n}{dz^n} P_J(z) &\to -\frac{d^n}{dz^n}\, \frac{\pi}{\sin \pi\alpha}\, P_\alpha(-z)\\
&\to \frac{d^n}{dz^n}\, \frac{P_\alpha(-z)}{\Delta} \to \frac{d^n}{dz^n}\, \frac{(-z)^\alpha}{\Delta} \qquad (4.2)
\end{aligned}$$

where $\Delta = \alpha - 1$ is treated as small.[5]

The e^{J+} functions needed for the evaluation of (3.2b) are:

$$\begin{aligned}
&e^{J+}_{0\,0} = P_J\,, \quad e^{J+}_{0\,1} = -e^{J}_{1\,0} = \frac{P'_J}{J(J+1)} \cong \frac{P'_J}{2}\,,\\
&e^{J+}_{0\,2} = \frac{P''_J}{(J-1)j(J+1)(J+2)} \cong \frac{P''_J}{6(J-1)}\,,\\
&e^{J+}_{1\,1} = \frac{P'_J + zP''_J}{J(J+1)} \cong \frac{P'_J + zP''_J}{2}\,,\\
&e^{J+}_{1\,2} = \frac{2P''_J + zP'''_J}{J(J+1)(J-1)(J+2)} \cong \frac{2P''_J + zP'''_J}{6\,(J-1)}\,.
\end{aligned} \qquad (4.3)$$

The approximations hold near $J = 1$.

We find, from Eqs. (3.1), (3.2b), (4.1), (4.2), and (4.3) the f^{+}'s for the indicated helicity states:

$$\begin{aligned}
f^{+}_{0\,0;\frac12\,\frac12} &\cong 3\left(\frac{k}{2p}\right)^{\frac12} \xi_{0\,0}\xi_{\frac12\,\frac12}\left\{\left(\frac{-z}{\Delta}\right) + (z \to -z)\right\},\\
f^{+}_{1\,0;\frac12\,\frac12} &\cong 3\left(\frac{k}{2p}\right)^{\frac12} \xi_{1\,0}\xi_{\frac12\,\frac12}\left\{\left(-\frac{1}{\sqrt{2}\Delta}\right) - (z \to -z)\right\},\\
f^{+}_{1\,1;\frac12\,\frac12} &\cong 3\left(\frac{k}{2p}\right) \xi_{1\,1}\xi_{\frac12\,\frac12}\left\{\left(-\frac{z}{\Delta}\right) + (z \to -z)\right\},\\
f^{+}_{1\,-1;\frac12\,\frac12} &\cong 3\left(\frac{k}{2p}\right)^{\frac12} \xi_{1\,-1}\xi_{\frac12\,\frac12}\left\{\left(\frac{-1}{\sqrt{6}z}\right) + (z \to -z)\right\},\\
f^{+}_{0\,0;\frac12\,-\frac12} &\cong 3\left(\frac{k}{2p}\right)^{\frac12} \xi_{0\,0}\xi_{\frac12\,-\frac12}\left\{\left(\frac{1}{\sqrt{2}\Delta}\right) - (z \to -z)\right\},\\
f^{+}_{1\,0;\frac12\,-\frac12} &\cong 3\left(\frac{k}{2p}\right)^{\frac12} \xi_{1\,0}\xi_{\frac12\,-\frac12}\left\{\left(\frac{-1}{2\Delta}\right) - (z \to -z)\right\},\\
f^{+}_{1\,1;\frac12\,-\frac12} &\cong 3\left(\frac{k}{2p}\right)^{\frac12} \xi_{1\,1}\xi_{\frac12\,-\frac12}\left\{\left(\frac{1}{\sqrt{2}\Delta}\right) - (z \to -z)\right\},\\
f_{1\,-1;\frac12\,-\frac12} &\cong 3\left(\frac{k}{2p}\right)^{\frac12} \xi_{1\,-1}\xi_{\frac12\,-\frac12}\left\{\left(\frac{-1}{2\sqrt{3}z}\right) + (z \to -z)\right\}.
\end{aligned} \qquad (4.4)$$

We now try to "measure" the products of ξ's oc-

[5] A numerical factor has been dropped in going from $P_\alpha(-z)$ to $(-z)^\alpha$; one can imagine incorporating it into the ξ's.

curring in (4.4) by performing the second-order calculation of the $N + \overline{N} \to \gamma + \gamma$ at high z. The relevant diagrams are shown in Fig. 1. The results of

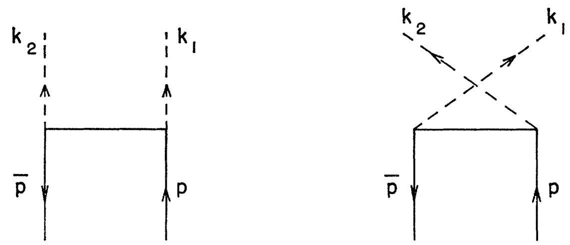

Fig. 1. Second-order diagrams for the reaction $N + \overline{N} \to \gamma + \gamma$.

the entirely straightforward second-order calculation are

$$
\begin{aligned}
f^{+}_{0\,0;\frac12\,\frac12} &\cong \gamma^2 \left(\frac{k}{p}\right)^{\frac12} \frac{m}{4\pi W}\left[\frac{-2\lambda^2}{E^2k}z + (z\to -z)\right],\\
f^{+}_{1\,0;\frac12\,\frac12} &\cong \gamma^2 \left(\frac{k}{p}\right)^{\frac12} \frac{m}{4\pi W}\left[-\frac{\sqrt{2}\lambda}{Ek} - (z\to -z)\right],\\
f^{+}_{1\,1;\frac12\,\frac12} &\cong \gamma^2 \left(\frac{k}{p}\right)^{\frac12} \frac{m}{4\pi W}\left[-\frac{z}{k} + (z\to -z)\right],\\
f^{+}_{1\,-1;\frac12\,\frac12} &\cong \gamma^2 \left(\frac{k}{p}\right)^{\frac12} \frac{m}{4\pi W}\left[-\frac{1}{kz} + (z\to -z)\right],\\
f^{+}_{0\,0;\frac12\,-\frac12} &\cong \gamma^2 \left(\frac{k}{p}\right)^{\frac12} \frac{m}{4\pi W}\left[\frac{2\lambda^2}{mEk} - (z\to -z)\right],\\
f^{+}_{1\,0;\frac12\,-\frac12} &\cong \gamma^2 \left(\frac{k}{p}\right)^{\frac12} \frac{m}{4\pi W}\left[\frac{-\sqrt{2}\lambda}{mk} - (z\to -z)\right],\\
f^{+}_{1\,1;\frac12\,-\frac12} &\cong \gamma^2 \left(\frac{k}{p}\right)^{\frac12} \frac{m}{4\pi W}\left[\frac{E}{mk} - (z\to -z)\right],\\
f^{+}_{1\,-1;\frac12\,-\frac12} &\cong \gamma^2 \left(\frac{k}{p}\right)^{\frac12} \frac{m}{4\pi W}\left[-\frac{E}{mkz} + (z\to -z)\right],
\end{aligned}
\tag{4.5}
$$

where in each f^+ the first term comes from the direct graph and the second (designated by $z \to -z$) come from the crossed graph.

We find, by comparing Eqs. (4.4) and (4.5) the following results for the product of ξ's:

$$
\begin{aligned}
\xi_{0\,0}\xi_{\frac12\,\frac12} &= A(2\lambda^2\Delta/E^2k)\,,\\
\xi_{0\,0}\xi_{\frac12\,-\frac12} &= A(2\sqrt{2}\lambda^2\Delta/mEk)\,,\\
\xi_{1\,0}\xi_{\frac12\,\frac12} &= A(2\lambda\Delta/Ek)\,,\\
\xi_{1\,0}\xi_{\frac12\,-\frac12} &= A(2\sqrt{2}\lambda\Delta/mk)\,,\\
\xi_{1\,1}\xi_{\frac12\,\frac12} &= A(\Delta/k)\,,\\
\xi_{1\,1}\xi_{\frac12\,-\frac12} &= A(\sqrt{2}\Delta E/mk)\,,\\
\xi_{1\,-1}\xi_{\frac12\,\frac12} &= A\sqrt{6}/k)\,,\\
\xi_{1\,-1}\xi_{\frac12\,-\frac12} &= A(\sqrt{2}\sqrt{6}E/mk)\,,
\end{aligned}
\tag{4.6}
$$

where $A = \gamma^2 m/4\pi W$.

Unfortunately, in this second-order calculation, all of the entries in Eq. (4.5) are in fact zero, so that the term by term identification just completed may be meaningless for all but the two nonsense amplitudes $f^{+}_{1\,-1;\frac12\,\pm\frac12}$. The point here is that the $1/z$ in Eq. (4.5) actually appears as the high z limit of a positive t pole, $(m^2 - t)^{-1}$ and the term $-1/z$ as the limit of a negative t pole, $(m^2 - u)^{-1}$ which can therefore be uniquely identified with the two terms in the asymptotic expressions for $f^{+}_{1\,-1;\frac12\,\pm\frac12}$ given in Eq. (4.4). Therefore only the last two of Eqs. (4.6), namely $\xi_{1\,-1}\xi_{\frac12\,\pm\frac12}$, are surely correct. The others must be regarded as conjectural, to be verified in the next order in Δ [since such terms, as our formulas predict, involve $\Delta \ln(-z)$ and $\Delta \ln(z)$ which are distinguishable]. It will turn out that $\Delta \sim \gamma^4$, hence this entails a sixth-order calculation which is left as an exercise for the reader.

In spite of the dubious correctness of most of Eqs. (4.6), it is interesting to observe that they are consistent with the expected factoring: We find that in all cases

$$\xi_{\frac12\,-\frac12}/\xi_{\frac12\,\frac12} = \sqrt{2}E/m\,. \tag{4.7}$$

5. REGGEIZATION OF THE $N\overline{N}$ AND $\gamma\gamma$ SCATTERING AMPLITUDES

We note that the amplitudes for the process $N + \overline{N} \to N + \overline{N}$ may be written, according to Eq. (3.5) as

$$
\begin{aligned}
f^{+}_{\frac12\,\frac12;\frac12\,\frac12} &\cong \tfrac12 \sum_J (2J+1)F^{J+}_{\frac12\,\frac12;\frac12\,\frac12}e^{J+}_{0\,0} + (z\to -z)\,,\\
f^{+}_{\frac12\,-\frac12;\frac12\,\frac12} &\cong \tfrac12 \sum_J (2J+1)F^{J+}_{\frac12\,-\frac12;\frac12\,\frac12}e^{J}_{0\,1} - (z\to -z)\,,\\
f_{\frac12\,-\frac12;\frac12\,-\frac12} &\cong \tfrac12 \sum_J (2J+1)F^{J+}_{\frac12\,-\frac12;\frac12\,-\frac12}e^{J+}_{1\,1} - (z\to -z)\,.
\end{aligned}
\tag{5.1}
$$

We again assume that the various F^{J+} have a Regge pole in the neighborhood of $J = 1$ and write

$$
F^{J+}_{\frac12\,\frac12;\frac12\,\frac12} \cong \frac{\xi^2_{\frac12\,\frac12}}{J-\alpha}\,, \qquad F^{J+}_{\frac12\,-\frac12;\frac12\,\frac12} \cong \frac{\xi_{\frac12\,-\frac12}\xi_{\frac12\,\frac12}}{J-\alpha}\,,
$$

$$
F^{J+}_{\frac12\,-\frac12;\frac12\,-\frac12} \cong \frac{\xi^2_{\frac12\,-\frac12}}{J-\alpha}\,. \tag{5.2}
$$

We find, for large z,

$$
\begin{aligned}
f^{+}_{\frac12\,\frac12;\frac12\,\frac12} &\cong -\tfrac32\,\xi^2_{\frac12\,\frac12}\{z\ln(-z) + (z\to -z)\}\,,\\
f^{+}_{\frac12\,-\frac12;\frac12\,\frac12} &\cong -(3/2\sqrt{2})\xi_{\frac12\,\frac12}\xi_{\frac12\,-\frac12}\\
&\qquad \times\{\ln(-z) - (z\to -z)\}\,,\\
f^{+}_{\frac12\,-\frac12;\frac12\,-\frac12} &\cong -\tfrac34\,\xi^2_{\frac12\,-\frac12}\{\ln(-z) - (z\to -z)\}\,.
\end{aligned}
\tag{5.3}
$$

Finally we treat the one important amplitude for $\gamma + \gamma \to \gamma + \gamma$, namely,

$$f^{+}_{1\ -1;1\ -1} \cong \sum_J (2J+1) F^{J+}_{1\ -1;1\ -1} e^{J+}_{2\ 2} + (z \to -z) \,, \tag{5.4}$$

and with[6]

$$F^{J+}_{1\ -1;1\ -1} \approx \xi^2_{1\ -1}\Delta/(J-\alpha) \,, \tag{5.5}$$

and

$$e^{J+}_{2\ 2} = \frac{2P_J' + 4zP_J'' + (z^2+1)P_J}{(J-1)J(J+1)(J+2)} \cong (J-1)/6z \text{ near } J=1 \,, \tag{5.6}$$

we find

$$f^{+}_{1\ -1,1\ -1} \cong -\frac{\xi^2_{1\ -1}\Delta}{2z} + (z \to -z) \,. \tag{5.7}$$

6. UNITARITY IN LOWEST ORDER PERTURBATION THEORY

We know in practice that $\gamma\gamma$ scattering occurs in lowest order via intermediate $N\bar{N}$ states and that $N\bar{N}$ scattering occurs in lowest order via two γ states (the contribution of ordinary one and two γ potential scattering is easily seen to be negligible near $J = 1$, or put otherwise, at large z). We may therefore directly calculate the imaginary parts of the products $\xi_{\frac{1}{2}\ \pm\frac{1}{2}} \cdot \xi_{\frac{1}{2}\ \pm\frac{1}{2}}$ and $\xi^2_{1\ -1}$ using our previous "measurements" of $\xi_{1\ -1}\xi_{\frac{1}{2}\ \pm\frac{1}{2}}$, Eq. (4.6).

Consider first the $\gamma\gamma$ scattering process. The unitarity relation for $F^{J+}_{1\ -1;1\ -1}$ takes the form [in lowest order (4th) perturbation theory]

$$\operatorname{Im} F^{J+}_{1\ -1;1\ -1} = \textstyle\sum_i pF^{J+}_{1\ -1;i}(F^{J+}_{1\ -1;i})^* \,, \tag{6.1}$$

where i stands for the two $N\bar{N}$ states, $|\frac{1}{2}\ \frac{1}{2}\rangle_+$ and $|\frac{1}{2}\ -\frac{1}{2}\rangle_+$. Then in the neighborhood of $J = 1$,

$$\operatorname{Im}\,[\Delta\xi^2_{1\ -1}/(J-\alpha)] = (p\xi^2_{1\ -1} \textstyle\sum_i \xi_i^2)/(J-1) \,, \tag{6.2}$$

or

$$\operatorname{Im}\,(\Delta\xi^2_{1\ -1}) = p\xi^2_{1\ -1} \textstyle\sum_i \xi_i^2 \,, \tag{6.3}$$

in lowest order. We have used the fact that the product $\xi_{1\ -1}\xi_i$ is real as shown by Eq. (4.6). Similarly, for the $N\bar{N}$ process, since[7] the sense amplitudes are all of higher order in γ^2,

$$\operatorname{Im} F_{i;j} = kF^{J+}_{i;1\ -1}(F^{J+}_{j;1\ -1})^* \,, \tag{6.4}$$

which leads to

$$\operatorname{Im}\,(\xi_i\xi_j) = k(\xi^2_{1\ -1}\xi_i\xi_j) \,. \tag{6.5}$$

Next we remark that the quantity $X_{ij} \equiv \xi_i\xi_j$ is, according to Eq. (5.3), measurable as the coefficient of $z \ln(-z)$ [or $\ln(-z)$] in $N\bar{N}$ scattering at high z. Its imaginary part, we see, is calculable from the $N + \bar{N} \to \gamma + \gamma$ experiment performed in Section 4. Similarly, we see that $Y \equiv \xi^2_{1\ -1}\Delta$ is calculable as the coefficient of z^{-1} in $\gamma\gamma$ scattering. Its imaginary part is also obtained from the $N + \bar{N} \to \gamma + \gamma$ process, according to Eq. (6.3). We note that the trajectory Δ may also be expressed in terms of these various quantities:

$$\Delta = \frac{\xi^2_{1\ -1}\xi_i\xi_j}{(\xi_{1\ -1}\xi_i)(\xi_{1\ -1}\xi_j)} = \frac{YX_{ij}}{\xi_{1\ -1}\xi_i\xi_j} = \frac{Y_iX_{ii}}{(\xi_{1\ -1}\xi_i)^2} \,. \tag{6.6}$$

As we have said, Y must be obtained from the 4th order $\gamma\gamma$ scattering and X_{ij} from 4th-order $N\bar{N}$ scattering; $\xi_{1\ -1}\xi_i$ is known from $N + \bar{N} \to \gamma + \gamma$.

We see from Eq. (6.6) that Δ is of order γ^4 since Y and X_{ij} are both $\sim\gamma^4$ and $\xi_{1\ -1}\xi_i \sim \gamma^2$. This implies that $\xi_{1\ -1}$ is of order γ^0 and ξ_i of order γ^2.

7. GENERALIZED UNITARITY

In the previous section we applied the two particle unitarity condition at $J = 1$ to lowest order of the coupling constant. We now wish to extend these considerations by studying the unitarity equations at $J = \alpha$ and show that our Regge pole assumptions, embodied in Eqs. (4.1), (5.2), and (5.5), are in fact consistent to all orders in the coupling constant. The complete unitarity equations are

$$\begin{aligned} \operatorname{Im} F_{ij} &= kF_{i\rho}F^*_{j\rho} + p \textstyle\sum_l F_{il}F^*_{jl} \,, \\ \operatorname{Im} F_{i\rho} &= kF_{i\rho}F^*_{\rho\rho} + p \textstyle\sum_l F_{il}F^*_{\rho l} \,, \\ \operatorname{Im} F_{\rho\rho} &= kF_{\rho\rho}F^*_{\rho\rho} + p \textstyle\sum_l F_{\rho l}F^*_{\rho l} \,, \end{aligned} \tag{7.1}$$

where i,j,l stand for the $N\bar{N}$ states $|\frac{1}{2}\ \frac{1}{2}\rangle_+$ and $|\frac{1}{2}\ -\frac{1}{2}\rangle_+$, ρ for the $\gamma\gamma$ nonsense state $|1\ -1\rangle_+$, and we have dropped the superscript $J+$. We have not included contributions from intermediate $\gamma\gamma$ sense states, since when we study Eqs. (7.1) in the neighborhood of $J = \alpha$ and expand the residues and the trajectory in powers of γ, one finds that sense state contributions are down by γ^4. It is important to note that our procedure is based on writing down *exact* unitarity equations, studying them in the neighborhood of $J = \alpha = 1 + \Delta$ and only at this point expanding the ξ's and Δ in powers of γ. The fact that

---

[6] This way of writing the Regge pole part of $F^{J+}_{1\ -1;1\ -1}$ is consistent with the assumption that the trajectory chooses "sense" at $\Delta = 0$, since the residue of a "nonsense" amplitude must vanish at such a point (see Ref. 2, Appendix). The Regge amplitudes for sensible 2γ channels would not be written with a $\Delta^{\frac{1}{2}}$ in the factored residue.

[7] This follows from the fact that for the sensible amplitudes there is nothing to cancel the $\sin \pi\Delta$ in the Regge pole contribution to scattering amplitude since the pole is really there. Consequently if we write $\xi_\sigma\xi_{\sigma'}/(J-\alpha)$ for a F^{J+}, we must have $(\xi_\sigma\xi_\sigma/\Delta) \sim \gamma^4$ which as we shall see in a moment leads to $\xi \sim \gamma^4$.

these quantities are being expanded in γ does not mean that we are not checking the unitarity relation to all orders of perturbation theory. We shall be able to *compute* only the leading terms in the expansions of ξ and Δ but will be able to show that exact condition implied by Eqs. (7.1) is in fact satisfied by these lowest order quantities.

All three of Eq. (7.1) are equivalent to the single relation (or its complex conjugate)

$$\operatorname{Im}\Delta = \operatorname{Im}\alpha = p\sum_j \xi_j^*\xi_j^* + k\xi_p^2\Delta = p\sum_j X_{jj}^* + kY. \tag{7.2}$$

In deriving this result we have assumed only that the product $\xi_p\xi_j$ is real. We must now check to see if our perturbation theoretic result, namely,

$$\alpha - 1 = Y\sum_i X_{ii}/\sum_i (\xi_i\xi_p)^2, \tag{7.3}$$

satisfies this general condition. We have (since $\xi_i\xi_p$ is real)

$$\begin{aligned}\operatorname{Im}\Delta &= \frac{Y\sum_i \operatorname{Im} X_{ii} + (\sum_i X_{ii}^*)\operatorname{Im} Y}{\sum_i (\xi_i\xi_p)^2} \\ &= \frac{k\xi_p^2\sum_i (\xi_i\xi_p)^2 + p\sum_i \xi_i^*\xi_i^*\sum_j (\xi_j\xi_p)^2}{\sum (\xi_i\xi_p)^2} \\ &= k\xi_p^2 + p\sum_i \xi_i^*\xi_i^*,\end{aligned} \tag{7.4}$$

which agrees with Eq. (7.2).

This completes the formal part of our work and we must now turn to experiment to see how things come out.

8. FOURTH-ORDER MEASUREMENTS AND CONCLUSIONS

The final step in our procedure is the evaluation of the 4th-order processes: $N\bar{N}$ and $\gamma\gamma$ scattering. We must check the factorization of the $N+\bar{N}\to N+\bar{N}$ amplitude in 4th order since our unitarity calculation has only verified the factorization of the imaginary part. Of course we need the quantity X_{ij} for the evaluation of Δ, Eq. (7.3), and in addition we need Y, associated with the $\gamma\gamma$ scattering.

The $N+\bar{N}\to N+\bar{N}$ amplitude is given, at large z, by the graphs shown in Fig. 2, where the positive t cut comes from 2a and the negative t cut from 2b. The contribution to the helicity amplitude (i.e. the conventional Jacob and Wick one) from the first graph (2a) is, at high z,

$$\begin{aligned} f^R_{\lambda_c\lambda_d;\lambda_a\lambda_b} &= \frac{m}{2\pi W}\ln(-z)\frac{\gamma}{8\pi^2}(\bar{U}_{-\lambda_d}\gamma V_{\lambda_c})(\bar{V}_{\lambda_a}\gamma U_{-\lambda_b}) \\ &\quad\times\int_{4\lambda^2}^{\infty}\frac{ds'}{k'W'}\,\frac{1}{s'-s-i\epsilon},\end{aligned} \tag{8.1}$$

where U and $\bar{V}$ are the initial nucleon and antinucleon states, respectively, and $\bar{U}$ and V the final ones. The mass of the vector boson (γ) is λ and

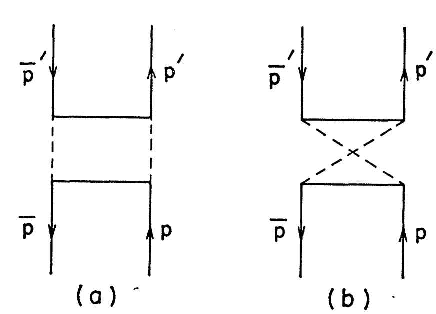

FIG. 2. Fourth-order diagrams for the reaction $N+\bar{N}\to N+\bar{N}$.

$s = W^2$. We now form the appropriate combinations of the several f^R to obtain f^{+R} and find

$$\begin{aligned} f^{+R}_{\frac12\frac12;\frac12\frac12} &\cong -(4m^2z/2\pi W)\ln(-z)\gamma^4 I_0(s), \\ f^{+R}_{\frac12-\frac12;\frac12\frac12} &\cong -(4Em/2\pi W)\ln(-z)\gamma^4 I_0(s), \\ f^{+R}_{\frac12-\frac12;\frac12-\frac12} &\cong -(4E^2/2\pi W)\ln(-z)\gamma^4 I_0(s),\end{aligned} \tag{8.2}$$

where

$$I_0(s) = \frac{1}{16\pi^2}\int_{4\lambda^2}^{\infty}\frac{ds'}{k'W'}\,\frac{1}{s'-s-i\epsilon}. \tag{8.3}$$

This result confirms the predictions of the Regge hypothesis in so far as powers of z and $\ln(-z)$ are concerned and also the factorization implied by Eqs. (4.7) and (6.5). In particular we see that the quantity $X_{\frac12\frac12;\frac12\frac12}$ is given by

$$X_{\frac12\frac12\frac12;\frac12} = \xi^2_{\frac12\frac12} = (4m^2/3\pi W)\gamma^4 I_0(s), \tag{8.4}$$

and $\xi_{\frac12-\frac12}\xi_{\frac12\frac12}$ and $\xi^2_{\frac12-\frac12}$ are given by the previously noted factorization, Eq. (4.7).

We turn finally to the evaluation of the quantity Y which involves the calculation of the $\gamma+\gamma\to\gamma+\gamma$ scattering amplitude. The graphs for this process are shown in Fig. 3. One of the complicating

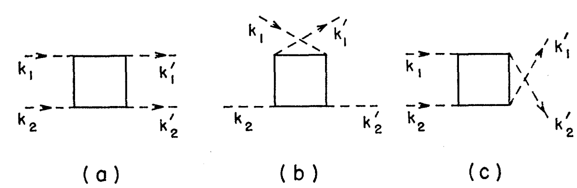

FIG. 3. Fourth-order diagrams for the reaction $\gamma+\gamma\to\gamma+\gamma$.

features of this calculation is the fact that the graph 3b has both a right- and left-hand z cut and these must be separated before taking the large z limit.

This can be done by restricting the range of integrations over the Feynman parameters in a reasonably straightforward manner. The diagram 3c has only a negative z cut and is uninteresting. Another difficulty is that although the predicted large z dependence of the amplitude $f^{\dagger}_{1\ -1;1\ -1}$ is $\sim 1/z$, the natural order of the various graphs is $\ln(-z)$ and consequently one is not involved with just the computation of the dominant contribution from the various graphs. It is unfortunately the case that we have not yet been given sufficient "running time" to have completed the measurement of the requisite $\gamma\gamma$ scattering amplitude. Consequently we cannot report the evaluation of the trajectory.

It is not profitable to speculate about the outcome of the calculation at any great length. We have seen that the idea of having a vacuum trajectory generated by the exchange of two massive vector bosons is consistent with elastic unitarity. One of the more interesting questions is to locate the place where $\Delta = \alpha - 1$ goes through zero. We can, of course, calculate Im Δ from unitarity as we have shown, but it is precisely the unknown subtraction question that forces our $\gamma\gamma$-scattering experiment. Perhaps the nicest result would be for Δ to be zero at $W^2 = 0$, as the fabled Pomeranchuk trajectory is supposed to behave. We have verified that as the boson mass goes to zero this is indeed the case. It could of course also be that Δ goes to zero at $W^2 = 0$ only when the coupling gets strong, or perhaps for some special value of the mass ratio, λ/m, other than the zero value we have mentioned.

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High-Energy Proton–Proton Scattering

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The optical model for high-energy proton–proton scattering which has been proposed in an earlier paper[1] predicts that, for large momentum transfers, the dependence of the elastic cross section on the square of the momentum transfer is approximately an inverse sixth power law. This prediction is borne out very well by the new measurements of Cocconi *et al.*[2] for their highest proton energies, near 30 BeV. The prediction has been checked down to cross sections as small as 2×10^{-12} of the forward scattering cross section and to center-of-mass scattering angles as large as 82°. For lower energies the measured cross sections deviate from the theoretical curve, becoming larger as the center-of-mass scattering angle approaches 90°.

The cross section for large momentum transfers depends on the behavior of the absorptive potential near $r = 0$, while that for small momentum transfers depends on the behavior for large r. A potential can be constructed to fit the observations for the entire range of momentum transfers. This is of Yukawa form for $r < 0.33 \times 10^{-13}$ cm, and of Gaussian form for $r > 1.1 \times 10^{-13}$ cm. The range of the Yukawa potential is determined by the width of the diffraction curve for large momentum transfer, the range of the Gaussian by the width for momentum transfer near zero.

The general features of high-energy elastic proton–proton scattering with large momentum transfer have been explained[1] in terms of a simple optical model. The change in wave number in the region of interaction was described by an absorptive potential,

$$k' - k = iV(r) \tag{1}$$

and V was supposed to be of Yukawa form,

$$V(r) = \eta e^{-\Lambda r}/r\,. \tag{2}$$

This leads to a cross section formula

$$(1/k^2)d\sigma/d\Omega = (1/\Lambda^4)F(t/\Lambda^2)^2\,, \tag{3}$$

with t the square of the momentum transfer. Numeri-

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